

MONOTONE INTERACTION OF WALK AND GRAPH: RECURRENCE VERSUS TRANSIENCE

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ABSTRACT. We consider recurrence versus transience for models of random walks on domains of \mathbb{Z}^d , in which monotone interaction enforces domain growth as a result of visits by the walk (or probes it sent), to the neighborhood of domain boundary.

1. INTRODUCTION

There has been much interest in studies of random walks in random environment (see [HMZ]). Of particular challenge are problems in which the walker affects its environment, as in reinforced random walks. In this context even the most fundamental question of recurrence versus transience is often open. For example, the recurrence of two dimensional linearly reinforced random walk with large enough reinforcement strength has been recently solved in [ACK], [ST]. The corresponding question by M. Keane for once reinforced random walk remains open. Moving to \mathbb{Z}^d , $d \geq 3$, the recurrence of once reinforced random walk is conjectured to be sensitive to the strength of the reinforcement (see e.g. [K1]). We consider here certain time-varying, highly non-reversible evolutions. Specifically, similarly to [DHS] we study discrete time simple random walk (SRW) $\{X_t\}$ on connected graphs $\mathbb{G}_t \uparrow \mathbb{G}_\infty \subseteq \overline{\mathbb{G}}$ (for some given, locally finite, connected graph $\overline{\mathbb{G}}$, adopting the notation \mathbb{D}_t in case $\overline{\mathbb{G}} = \mathbb{Z}^d$). That is, starting at given \mathbb{G}_0 and initial site $X_0 \in \mathbb{G}_0$, the sequence $\{X_t\}$ is adapted to some filtration $\{\mathcal{F}_t\}$ with $\{\mathbb{G}_t\}$ being \mathcal{F}_t -previsible (most often using $\mathcal{F}_t = \sigma(X_s, s \leq t)$ the canonical filtration of the SRW), and having $X_t = x$, one chooses X_{t+1} uniformly among all neighbors of x within \mathbb{G}_t .

Our companion paper [DHS] deals with $\{\mathbb{G}_t\}$ growing independently of $\{X_t\}$, a situation in which universality is to be expected (c.f. Conjectures 1.1-1.3 of [DHS] and the analogous conjectures made in [ABGK] for the corresponding strictly positive and finite conductances model). In contrast, rich and often counter intuitive behavior occurs when focusing on genuine monotone interaction between the path $\{X_0, \dots, X_t\}$ of the walk and the growth $\mathbb{G}_{t+1} \setminus \mathbb{G}_t$ of the graphs. In this context, our Lemma 1.3 provides an equivalent condition for transience/recurrence of the SRW on growing $\{\mathbb{G}_t\}$. We examine here its consequences for various monotone interactions. In particular, in *open by touch* type interactions a local or bounded number of edges is added to \mathbb{G}_t as result of each visit to its boundary sites (see

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Defn. 1.5). We then expect the SRW on \mathbb{G}_t to inherit the transience of $\overline{\mathbb{G}}$ when starting at large enough \mathbb{G}_0 (see Prop. 1.7 and Conjecture 1.10), while it should be recurrent when \mathbb{G}_0 is small and \mathbb{G}_t almost regular (see Defn. 1.14 and Prop. 1.15, but beware the counter example of transience provided in Prop. 1.18). This recurrence should be related to such interaction requiring order of surface-area visits to the graph's boundary. In the same direction we find sharp transition between transience and recurrence at lower than surface-area growth of the number of boundary visits for *expanding glassy spheres* type interactions. These are of almost regular shape due to global growth upon completion of the required number of visits to the current graph's boundary, see Defn. 1.11 and Prop. 1.12. Finally, we consider *probing simple random walk* where a variable/fixed number of guided/unguided probes is sent from walker's current location, with each probe adding a site at the graph's boundary. In this setting guided probes may flip the walk between transience and recurrence, whereas for unguided probes the SRW supposedly inherits the transience/recurrence of the underlying graph \mathbb{Z}^d (see Prop. 1.19 and Conjecture 1.21).

Recall that for any time-homogeneous Markov chain $\{Z_t\}$ on countable state space $\overline{\mathbb{G}}$, a zero-one law applies for the recurrence of state $z \in \overline{\mathbb{G}}$, namely for the event $\{N_z = \infty\}$ and $N_z := \sum_t \mathbb{I}_{\{z\}}(Z_t)$. Further, such recurrence, i.e. $\mathbb{P}_z(N_z = \infty) = 1$, is equivalent to $\mathbb{E}_z(N_z) = \infty$ and to $\mathbb{P}_z(Z_t = z \text{ for some } t) = 1$. In contrast, neither such equivalence, nor zero-one law apply in our more general setting of monotonically interacting SRW on growing graphs. For example, both zero-one law and equivalence break for suitable choices of $\{1, \dots, \infty\}$ -valued random variable K , taking \mathbb{G}_t , $t \leq K$ a single edge adjacent to the origin and $\mathbb{G}_t = \mathbb{Z}^3$ for $t > K$. This prompts our selection hereafter of the following definition of sample-path recurrence.

Definition 1.1. A site $x \in \overline{\mathbb{G}}$ is recurrent for the sample path of SRW $\{X_t\}$ on $\{\mathbb{G}_t\}$ if $\{X_t = x \text{ i.o.}\}$. Otherwise we say that the site $x \in \overline{\mathbb{G}}$ is transient for this sample path of the SRW on $\{\mathbb{G}_t\}$.

Let $\deg_{\mathbb{G}}(z)$ denote the degree of vertex z in graph \mathbb{G} , $d^{\mathbb{G}}(x, y)$ the graph distance in \mathbb{G} between $x, y \in \mathbb{G}$ and $\mathbb{B}^{\mathbb{G}}(z, r)$ the corresponding (closed) ball of radius r and center z in $(\mathbb{G}, d^{\mathbb{G}})$, with \mathbb{B}_r denoting the projection on \mathbb{Z}^d of the (closed) Euclidean ball of radius r centered at the origin. Hereafter, we set $X_0 = 0$, assuming that $\mathbb{B}^{\overline{\mathbb{G}}}(0, 1) \subset \mathbb{G}_0$, and in view of the following lemma, focus without loss of generality on sample path recurrence of this distinguished vertex.

Lemma 1.2. For any SRW $\{X_t\}$ on monotone increasing connected graphs $\{\mathbb{G}_t\}$, on the event $\{X_t = 0, \text{ f.o.}\}$ of transience of 0 we have that a.s. $d^{\mathbb{G}_t}(0, X_t) \rightarrow \infty$ when $t \rightarrow \infty$. Consequently, with probability one, 0 is transient for the sample path of the SRW if and only if every vertex of $\overline{\mathbb{G}}$ is transient for this path.

Proof. Fixing r finite, let $A_t = \{X_{t+u} = 0, \text{ some } u \geq 0\}$ and $\Gamma_{t,r} = \{d^{\mathbb{G}_t}(0, X_t) \leq r\}$. Since $X_0 = 0$, and $\mathbb{G}_t \subseteq \overline{\mathbb{G}}$ are non-decreasing (so in particular $\mathbb{B}^{\mathbb{G}_t}(0, r) \subseteq \mathbb{B}^{\overline{\mathbb{G}}}(0, r)$, for any $t \geq 0$), and $\overline{\mathbb{G}}$ is locally finite, it follows that

$$\mathbb{E}_{X_t}(\mathbb{I}_{A_t} | \mathcal{F}_t) \geq M_r^{-r} \mathbb{I}_{\Gamma_{t,r}},$$

with $M_r := \max_{z \in \mathbb{B}_{\overline{\mathbb{G}}}(0,r)} \{\deg_{\overline{\mathbb{G}}}(z)\}$ finite. When $t \rightarrow \infty$ we have that $\mathbb{I}_{A_t} \rightarrow \mathbb{I}_{\{X_t=0 \text{ i.o.}\}}$ and

$$\liminf_{t \rightarrow \infty} \mathbb{I}_{\Gamma_{t,r}} = \mathbb{I}_{\{X_t \in \mathbb{B}^{\mathbb{G}_t}(0,r) \text{ i.o.}\}},$$

hence by Lévy's upward theorem, w.p.1. if $\{X_t \in \mathbb{B}^{\mathbb{G}_t}(0,r) \text{ i.o.}\}$ then $\{X_t = 0 \text{ i.o.}\}$. Taking $r \rightarrow \infty$ we deduce that w.p.1. transience at 0 of the sample path implies finitely many visits of X_t to $\mathbb{B}^{\mathbb{G}_t}(0,r)$ for each r , hence both transience of every $x \in \overline{\mathbb{G}}$ for this sample path and that $d^{\mathbb{G}_t}(0, X_t) \rightarrow \infty$ when $t \rightarrow \infty$. \square

Next, with $\deg_{\overline{\mathbb{G}}}(z)$ the maximal possible degree of vertex z , we define the boundary set

$$\partial \mathbb{G}_t := \{z \in \mathbb{G}_t : \deg_{\mathbb{G}_t}(z) < \deg_{\overline{\mathbb{G}}}(z)\},$$

of \mathbb{G}_t (consisting of all vertices of \mathbb{G}_t whose degree may yet change as $\mathbb{G}_t \uparrow \mathbb{G}_\infty$), and characterize transience via summability of $p_n := \mathbb{P}(A_n | \mathcal{F}_{\eta_n})$, for events

$$A_n := \{\exists s \in [\eta_n, \sigma_n) : X_s = 0\},$$

and the following \mathcal{F}_t -stopping times $\{\eta_n, \sigma_n\}$, starting at $\eta_0 = 0$:

$$\begin{aligned} \sigma_n &:= \inf\{t \geq \eta_n : X_t \in \partial \mathbb{G}_{\eta_n}\}, \quad n \geq 0 \\ \eta_{n+1} &:= \inf\{t \geq \sigma_n : X_t \notin \partial \mathbb{G}_t\}. \end{aligned}$$

Lemma 1.3. *Let $S := \sum_n p_n$.*

- (a) *The sample path of $\{X_t\}$ is a.s. recurrent on $S = \infty$;*
- (b) *Conversely, if SRW on the fixed graph $\overline{\mathbb{G}}$ is transient, then the sample path of $\{X_t\}$ is a.s. transient on $S < \infty$.*

Remark 1.4. In [ABGK, Sections 4,5] it is shown that a monotonically interacting strictly positive and finite conductance model on a tree $\overline{\mathbb{G}}$ tends to follow the recurrence/transience of its starting and ending conductances (in particular, this applies for $\overline{\mathbb{G}} = \mathbb{Z}$). However, this approach, based on using flows to construct suitable sub or super martingales, is limited in scope to trees (indeed [ABGK, Section 6] provides a counter example to such conclusion in case $\overline{\mathbb{G}} = \mathbb{Z}^2$). In contrast, while less explicit, Lemma 1.3 applies for any $\overline{\mathbb{G}}$. Further, the advantage of this lemma lies in p_n being the probability that a SRW on fixed graph $\overline{\mathbb{G}}$ starting at the random position X_{η_n} visits 0 before $\partial \mathbb{G}_{\eta_n}$, hence amenable to the use of classical hitting probability estimates for random walk on a fixed graph.

Proof. Recall Paul Lévy's extension of Borel-Cantelli lemma (see [Du, Theorem 5.3.2]), that a.s. $S = \infty$ if and only if $\{A_n \text{ i.o.}\}$ which immediately yields part (a). Further, $\deg_{\mathbb{G}_t}(0) = \deg_{\overline{\mathbb{G}}}(0)$ for all t (by our assumption that $\mathbb{B}^{\overline{\mathbb{G}}}(0,1) \subset \mathbb{G}_0$), hence $X_s \neq 0$ whenever $s \in [\sigma_n, \eta_n)$ and the a.s. transience of 0 for $\{X_t\}$ in case $S < \infty$ follows, provided $\sigma_n < \infty$ for all n . To rule out having with positive probability $\{\sigma_n = \infty \text{ and } X_t = 0 \text{ for infinitely many } t \geq \eta_n\}$, note that by our assumption of transience of the SRW on $\overline{\mathbb{G}}$, the former can not occur if $\partial \mathbb{G}_{\eta_n} = \emptyset$. So, assuming hereafter that $\partial \mathbb{G}_{\eta_n}$ is non-empty, conditional on \mathcal{F}_{η_n} , if the irreducible SRW on the fixed connected graph \mathbb{G}_{η_n} visits 0 i.o., then it a.s. would also enter $\partial \mathbb{G}_{\eta_n}$ in finite time, namely having $\sigma_n < \infty$. \square

Of particular interest to us are the *open by touch* type interaction models, in which graph growth occurs only upon the walker's visits of the graph's boundary sites.

Definition 1.5. We say that $Y_t \in \mathbb{G}_t \uparrow \mathbb{G}_\infty \subseteq \overline{\mathbb{G}}$ is an open by touch (OBT) interaction model, if $\mathbb{G}_{t+1} = \mathbb{G}_t$ except when $Y_t \in \partial\mathbb{G}_t$, at which times all edges of $\mathbb{B}^{\overline{\mathbb{G}}}(Y_t, 1)$ are added to \mathbb{G}_{t+1} . More generally, in a *partially open by touch* (POBT) interaction we add to \mathbb{G}_{t+1} , with uniformly bounded away from zero probability, one (or more) of the edges adjacent to $Y_t \in \partial\mathbb{G}_t$, in a *first open by touch* (FOBT) interaction such addition of edges adjacent to $x \in \partial\mathbb{G}_t$ occurs *only at the first visit* of x by Y_t , whereas in a *remotely open by touch* (ROBT) we only require that the collection of edges \mathbb{A}_t added to \mathbb{G}_t when $Y_t \in \partial\mathbb{G}_t$ be of uniformly bounded cardinality.

In any OBT model the walker opens with a one step delay all edges of $\overline{\mathbb{G}}$ adjacent to its current position, in effect performing SRW on $\overline{\mathbb{G}}$, except at her first visit to certain sites. Thus, one may expect that OBT interactions inherit the transience of SRW on $\overline{\mathbb{G}}$, as long as they follow that SRW update rule, except maybe when at $\partial\mathbb{G}_t$. Indeed, we utilize such notion of *extended simple random walks* when studying this question (in the sequel).

Definition 1.6. We say that a ROBT interaction model on $\mathbb{G}_t \uparrow \mathbb{G}_\infty \subseteq \overline{\mathbb{G}}$ forms an extended simple random walk if Y_t follows the steps of SRW on $\overline{\mathbb{G}}$ except for allowing whenever $Y_t \in \partial\mathbb{G}_t$ to have any \mathcal{F}_t -measurable mechanism for choosing $Y_{t+1} \in \mathbb{G}_t \cap \mathcal{C}(Y_t)$, for some fixed $\mathcal{C}(x) := \mathbb{B}^{\overline{\mathbb{G}}}(x, r(x))$, $c < 1$ and $1 \leq r(x) \leq cd^{\overline{\mathbb{G}}}(x, 0)$.

Note however that in this context, [K2] shows that for transient $\overline{\mathbb{G}} = \mathbb{Z}^d$, $d \geq 3$, starting at $Y_0 = 0$ and $\mathbb{G}_0 = \mathbb{B}(0, 1)$, it is possible to have an OBT extended simple random walk, whose sample path is a.s. recurrent at 0, even with $r(x) = 1$ everywhere. Specifically, this is done by creating drift to the origin at first visits, such that $\mathbb{E}[Y_{t+1} - Y_t | \mathcal{F}_t] = -\delta Y_t / \|Y_t\|_1$ for some $\delta > 0$ and all $Y_t \in \partial\mathbb{G}_t$. In contrast, with Lemma 1.3 applicable for extended simple random walks, we next prove that 0 is a.s. transient for the sample path in any POBT starting with $\mathbb{D}_0 \subseteq \mathbb{Z}^d$, $d \geq 3$ of fast enough diminishing density of closed edges.

Proposition 1.7. *If SRW on $\overline{\mathbb{G}}$ is transient, all sites (but not all edges), of $\overline{\mathbb{G}}$ are in \mathbb{G}_0 and*

$$S_\star := \sum_{x \in \partial\mathbb{G}_0} \sup_{y \in \mathcal{C}(x)} \{\mathbb{P}_y(\text{SRW on } \overline{\mathbb{G}} \text{ ever hits } 0)\} < \infty, \quad (1.1)$$

then almost every sample path in any OBT extended simple random walk Y_t on $\mathbb{G}_t \uparrow \mathbb{G}_\infty$ is transient. The same applies for any POBT provided $\overline{\mathbb{G}}$ is of uniformly bounded degrees. In particular, in case $\overline{\mathbb{G}} = \mathbb{Z}^d$, $d \geq 3$, denote by $N(k)$ the number of vertices in $\partial\mathbb{D}_0$ that are on the boundary of the box of side length k centered at 0. Then w.p.1. the sample path of POBT extended simple random walk Y_t on $\mathbb{D}_t \uparrow \mathbb{D}_\infty$ is transient, provided

$$\sum_{k=1}^{\infty} N(k)k^{2-d} < \infty. \quad (1.2)$$

Remark 1.8. There is no analog of Proposition 1.7 for recurrent $\overline{\mathbb{G}}$. For example, taking $\overline{\mathbb{G}} = \mathbb{Z}^2$ with $Y_0 = 0$ in connected \mathbb{G}_0 whose closed edges consist of exactly one among those touching sites $(\pm s_t, 0)$ and $(0, \pm s_t)$, for $s_t = \lfloor (1+c)^{t-1} \rfloor$, $t \geq 1$ and $c < 1$ (hence of fast diminishing density), one can create OBT extended simple random walk with only four sample path, $Y_t = (\pm s_t, 0)$ or $Y_t = (0, \pm s_t)$, all of which are transient.

Remark 1.9. Proposition 1.7 applies regardless of the manner and probability in which edges are added to $\partial\mathbb{G}_t$ in the POBT interaction, but this may be a somewhat delicate matter when starting with smaller graph \mathbb{G}_0 . For example, walking on sub-domains \mathbb{D}_t of the recurrent $\overline{\mathbb{G}} = \mathbb{Z}^2$, starting at $\mathbb{D}_0 = \mathbb{B}(0, 1)$, Proposition 1.18 proves a.s. transience in FOBT interaction for which only the right/up/down edges out of each site are added to \mathbb{D}_{t+1} (upon first visit to the site by the SRW on \mathbb{D}_t).

While Proposition 1.7 requires diminishing density of closed edges in $\mathbb{D}_0 \subseteq \mathbb{Z}^d$, $d \geq 3$, we believe that SRW with OBT interaction has a.s. transient sample path as soon as the open edges of \mathbb{D}_0 percolate in \mathbb{Z}^d . Specifically, we make the following conjecture.

Conjecture 1.10. *For the OBT interaction in \mathbb{Z}^d , $d \geq 3$, upon starting its SRW at $0 \in \mathbb{D}_0$, if \mathbb{D}_0 is the infinite cluster of bond/site super-critical percolation, then the corresponding sample-path is a.s. transient.*

Having seen the effect of the initial graph on recurrence versus transience for certain interacting walks and graphs, we turn to the implications of asymptotic regularity of \mathbb{G}_t . To this end, we first consider *expanding glassy spheres* interactions, in which growth requires certain number of visits by the walk to the graph's boundary, at which point a global expansion of the graph occurs.

Definition 1.11. Fix $c \geq 1$, $N(k) \geq 1$ and infinite (connected, locally finite), graph $\overline{\mathbb{G}}$, setting $\overline{\mathbb{B}}_k := \mathbb{B}^{\overline{\mathbb{G}}}(0, ck)$. The *expanding glassy spheres* (EGS) interaction consists of SRW Z_t on $\mathbb{G}_t = \overline{\mathbb{B}}_k$ for $t \in [\tau_k, \tau_{k+1})$, starting at $Z_0 = 0$ and with \mathcal{F}^Z -stopping times $\tau_1 := 0$,

$$\tau_{k+1} := \inf\{s > \tau_k : \sum_{t=\tau_k}^{s-1} \mathbb{I}_{\partial\overline{\mathbb{B}}_k}(Z_t) = N(k)\}, \quad k \geq 1.$$

Alternative sets may be used as well. For example, as the name EGS suggests, in case $\overline{\mathbb{G}} = \mathbb{Z}^d$, $d \geq 1$, we define such EGS $\{Z_t\}$ as being confined to the projected Euclidean ball \mathbb{B}_{ck} until making the prescribed number of visits $N(k)$ to its boundary, at which time this projected ball expands to $\mathbb{B}_{c(k+1)}$, and so on (instead of using the graph distance on \mathbb{Z}^d for defining such balls).

Employing Lemma 1.3 we determine the transition between recurrence and transience for EGS on \mathbb{Z}^d , $d \geq 2$ in terms of asymptotic growth of the prescribed hit counts $\{N(k)\}$ (showing in particular that for $d = 2$, such EGS is always recurrent).

Proposition 1.12. *For $\overline{\mathbb{G}}$ of bounded degrees and $\mathcal{C}_k := \{x \in \overline{\mathbb{B}}_k : d^{\overline{\mathbb{G}}}(x, \partial\overline{\mathbb{B}}_k) = 1\}$,*

$$\sum_{k=1}^{\infty} N(k) \inf_{x \in \mathcal{C}_k} \mathbb{P}_x(\text{SRW on } \overline{\mathbb{G}} \text{ hits } 0 \text{ before } \partial\overline{\mathbb{B}}_k) = \infty, \quad (1.3)$$

yields a.s. sample path recurrence for the corresponding EGS, whereas if

$$\sum_{k=1}^{\infty} N(k) \sup_{x \in \mathcal{C}_k} \mathbb{P}_x(\text{SRW on } \overline{\mathbb{G}} \text{ hits } 0 \text{ before } \partial\overline{\mathbb{B}}_{k+1}) < \infty, \quad (1.4)$$

then a.s. the sample path of the corresponding EGS is transient. In particular, for $\overline{\mathbb{G}} = \mathbb{Z}^d$, $d \geq 2$, we have zero-one law for transience/recurrence of the EGS interaction sample path which is a.s. transient if and only if

$$\sum_{k=1}^{\infty} N(k)k^{1-d} < \infty. \quad (1.5)$$

Remark 1.13. The assumed uniform bound on $\deg_{\overline{\mathbb{G}}}(z)$ is only required for getting the factor $N(k)$ within the sum on LHS of (1.3). To this end it suffices to have a uniform (in k and z), upper bound on the expected hitting time of \mathcal{C}_k by the SRW on $\overline{\mathbb{B}}_k$ starting at $z \in \partial\overline{\mathbb{B}}_k$. Some such condition is relevant for recurrence/transience of EGS. Indeed, consider $c = 1$ and $\overline{\mathbb{G}}$ arranged in layers, with each site in k -th layer (i.e. of distance k from 0), having $\ell_k^\pm = \ell_k q_k p_k^\pm \geq 1$ edges to $(k \pm 1)$ -th layer (with $\ell_0^- = 0$), and $\ell_k^0 = \ell_k(1 - q_k)p_k^-$ edges to its own layer. For EGS on such graph, $t \mapsto d_{\overline{\mathbb{G}}}^{\mathbb{G}}(0, Z_t)$ evolves up to holding times within layers, as a modified birth-death chain W_s on \mathbb{Z}_+ , starting at $W_0 = 0$, moving with probability p_k^\pm from k to $(k \pm 1)$, but opening edge $(k, k+1)$ only after an independent $\text{Binomial}(N(k), q_k)$ steps from k to $(k-1)$ are made by $\{W_s\}$. Conditions (1.3) and (1.4) amount to divergence and convergence, respectively, of $\sum_k N(k)\mathbb{P}_{k-1}(W_s \text{ hits } 0 \text{ before } k)$, which for uniformly bounded $\{q_k\}$ is indeed a sharp criterion for recurrence/transience of $\{W_s\}$ (and thereby of the EGS). However, the latter series is missing the factor q_k , so for unbounded $\{q_k\}$ it is often wrong for determining transience versus recurrence of $\{W_s\}$.

Building on the insight provided by Proposition 1.12, we relate the regularity, as defined below, of the graphs \mathbb{D}_t produced by an ROBT interaction model on \mathbb{Z}^d , $d \geq 2$, with the a.s. sample path recurrence for the corresponding SRW.

Definition 1.14. We say that \mathbb{K} is a γ -almost regular shape for growing domains $\{\mathbb{D}_t\}$ in \mathbb{Z}^d , $d \geq 2$, if $\mathbb{D}_t \supseteq f(t)\mathbb{K} \cap \mathbb{Z}^d$ for all t large and some non-decreasing $f(\cdot) \geq 1$, such that $d^{\mathbb{D}_t}(z, f(t)\mathbb{K}) \leq \gamma \log f(t)$ for all $z \in \mathbb{D}_t$.

Proposition 1.15. There exist $c_d > 0$, such that if $0 \in \mathbb{D}_0$ a finite connected domain in \mathbb{Z}^d , $d \geq 2$ and the ball \mathbb{B} is c_d -almost regular shape for the growing domains \mathbb{D}_t , then 0 is a.s. recurrent for the sample path of any SRW $\{R_t\}$ of ROBT interaction with \mathbb{D}_t .

Remark 1.16. If the range of an ROBT interaction model $\{R_s, s \leq t\}$ contains the whole $f(t)\mathbb{B}$ ball within \mathbb{D}_t , one may be tempted to conclude that recurrence of $\{R_t\}$ then trivially follows since every site of \mathbb{Z}^d is for sure being visited at least once. However, as mentioned before, unlike SRW on fixed graph, in case of growing domains the walk may nevertheless w.p.1 return to 0 only finitely many times. So, Proposition 1.15 provides a non-trivial conclusion, even in this setting.

Remark 1.17. For any $\mathbb{K} \subset \mathbb{R}^d$ and $f > 0$, let $(f\mathbb{K})_d = f\mathbb{K} \cap \mathbb{Z}^d$ denote the corresponding lattice projection. Proposition 1.15 then holds for any \mathbb{K} such that for some c finite,

$$\liminf_{f \rightarrow \infty} \inf_{\{z \in (f\mathbb{K})_d : d(z, \partial(f\mathbb{K})_d) \geq c \log f\}} \frac{1}{\log f} \log \mathbb{P}_z(\text{SRW hits } 0 \text{ before } \partial(f\mathbb{K})_d) > -d. \quad (1.6)$$

Transience w.p.1. is proved in [ABGK, Sect. 6] for the following monotone increasing conductance model on edges of \mathbb{Z}^2 : starting at $t = 0$ with walker at the origin and conductance 1 at each edge, upon walker's first visit of each vertex, the conductances of its adjacent edges to the right/up/down are increased to 2. Adapting the arguments of [ABGK, Sect. 6], we next provide examples of a.s. transient FOBT interaction between SRW $\{E_t\}$ and the corresponding growing domains $\{\mathbb{D}_t\}$ in \mathbb{Z}^2 , emphasizing the role of the initial graph \mathbb{D}_0 .

Proposition 1.18. *Consider the SRW $\{E_t\}$ on $\mathbb{D}_t \subseteq \mathbb{Z}^2$, that starts from $E_0 = 0 \in \mathbb{D}_0$ and opens only the three right/up/down edges adjacent to each site that it first visits (where after each such opening the walk stays put for one step before choosing its next position, now on \mathbb{D}_{t+1}).*

- (a) *If \mathbb{D}_0 consists of the vertices of \mathbb{Z}^2 with each edge of \mathbb{Z}^2 independently chosen to be in \mathbb{D}_0 with same probability $p \in [0, 1)$, then the sample path of E_t is \mathbb{P}_p -a.s. transient.*
- (b) *Alternatively, the sample path of $\{E_t\}$ is a.s. transient whenever $k^{-r}|\mathbb{D}_0 \cap [-k, k]^2| \rightarrow 0$ as $k \rightarrow \infty$, for some constant $r < 3/4$.*

Our final result deals with a.s. transience for the *probing simple random walk* (PSRW), $\{K_t\}$ on growing domains $\{\mathbb{D}_t\}$ in \mathbb{Z}^d , $d \geq 2$. Starting at $K_0 = 0$ and $\mathbb{D}_0 = \{0\}$, such PSRW is allowed to send at time t some \mathcal{F}_t -adapted number of probes $m(t)$, with each probe adding precisely one site to \mathbb{D}_t (and opening all relevant edges connecting those sites with the existing graph), prior to the walk's move from K_t to $K_{t+1} \in \mathbb{D}_{t+1}$. The aim of the PSRW is to guarantee a.s. transience of its sample path with minimal asymptotic running average number of probes $\bar{m}_t := t^{-1} \sum_{s=1}^t m(s)$. Conversely, the PSRW may aim at a.s. recurrence of its sample path with a maximal asymptotic running average number of probes. In different versions of this problem the PSRW may or may not have control on the probes locations and the number of probes being used in each step.

Proposition 1.19.

- (a) *For \mathbb{Z}^d , $d \geq 2$ and any $\epsilon > 0$, there exist \mathcal{F}_t -adapted $\{m(t)\}$ and choices of the $m(t)$ probe positions at \mathbb{Z}^d -distance-one from \mathbb{D}_t , such that eventually $\bar{m}_t < \epsilon$ and the sample path of K_t is a.s. transient. There also exist (some other) such probe numbers and locations for which eventually $\bar{m}_t > \epsilon^{-1}$ and the sample path of K_t is a.s. recurrent.*
- (b) *Suppose each probed site is chosen according to the hitting measure of \mathbb{D}_t^c by a SRW on \mathbb{Z}^d which starts at the current position K_t of the PSRW. Then there exist finite constants c_d and \mathcal{F}_t -adapted process $\{m(t)\}$ such that a.s. $\limsup_t \bar{m}_t < c_d$, the PSRW sample path is transient in case $d \geq 3$, and recurrent with $\liminf_t \bar{m}_t$ arbitrarily large, in case $d = 2$.*

Remark 1.20. Two obvious open problems are whether part (b) of Proposition 1.19 holds for any $c_d > 0$, $d \geq 3$, and whether in this context one can also select a process $m(t)$ yielding a.s. sample path recurrence when $d \geq 3$ and transience when $d = 2$.

We end with the following conjecture and related open problems.

Conjecture 1.21. *In the setting of part (b) of Proposition 1.19 there exist \mathcal{F}_t -adapted $m(t)$ which is uniformly bounded above by non-random integer λ_d , and a PSRW having a.s. transient sample path when $d \geq 3$, and a.s. recurrent sample path with $m(t) \geq 1$, when $d = 2$. If this conjecture is valid, does it apply for $\lambda_d = 1$ and does the same apply even for constant $m(t) = \lambda_d$ (i.e. removing all control from the PSRW)?*

2. PROOF OF PROPOSITIONS 1.7, 1.12 AND 1.15

Proof of Proposition 1.7. First consider an OBT extended simple random walk. Recall Remark 1.4 that p_n is the probability that SRW on $\overline{\mathbb{G}}$ starting at Y_{η_n} visits 0 before $\partial\mathbb{G}_{\eta_n}$, and Definition 1.6 that $Y_{\eta_n} \in \mathcal{C}(Y_{\eta_{n-1}})$. Hence, setting

$$g(x) := \sup_{y \in \mathcal{C}(x)} \{\mathbb{P}_y(\text{SRW on } \overline{\mathbb{G}} \text{ ever hits } 0)\},$$

we have that for any $n \geq 1$,

$$p_n \leq \mathbb{P}_{Y_{\eta_n}}(\text{SRW on } \overline{\mathbb{G}} \text{ ever hits } 0) \leq g(Y_{\eta_{n-1}}).$$

By assumption, all sites of $\overline{\mathbb{G}}$ are already in \mathbb{G}_0 , and with the OBT interaction enforcing that $Y_t \notin \partial\mathbb{G}_{t+1}$, the distinct sites $\{Y_{\eta_n}, n \geq 1\}$ are all in $\partial\mathbb{G}_0$. Hence, $S = \sum_{n \geq 1} p_n$ is bounded above by the assumed finite term S_* of (1.1) and the a.s. sample path transience of $\{Y_t\}$ follows by part (b) of Lemma 1.3. In case of an POBT interaction, the same derivation yields the bound

$$S \leq \sum_{x \in \partial\mathbb{G}_0} L_x g(x),$$

where L_x denotes the number of visits by $\{Y_t\}$ to $x \in \partial\mathbb{G}_0$, up to the possibly infinite stopping time $\theta_x := \inf\{t \geq 0 : \mathbb{B}^{\overline{\mathbb{G}}}(x, 1) \subseteq \mathbb{G}_t\}$. The POBT interaction adds at least one edge to \mathbb{G}_{t+1} upon each visit to $Y_t \in \partial\mathbb{G}_t$ with probability at least $\epsilon > 0$. Hence, $\mathbb{E}[L_x | \mathcal{F}_0] \leq \epsilon^{-1} \deg_{\overline{\mathbb{G}}}(x)$. In particular, almost surely, S is finite if

$$\mathbb{E}[S | \mathcal{F}_0] \leq \epsilon^{-1} \sum_{x \in \partial\mathbb{G}_0} \deg_{\overline{\mathbb{G}}}(x) g(x) < \infty,$$

which, for $\overline{\mathbb{G}}$ of uniformly bounded degrees, follows from finiteness of S_* .

Specializing to $\overline{\mathbb{G}} = \mathbb{Z}^d$, $d \geq 3$, of uniformly bounded degree, recall Definition 1.6 that here $\|y\|_1 \geq \|x\|_1 - \|x - y\|_1 \geq (1 - c)\|x\|_1$ for any $y \in \mathcal{C}(x)$. Thus, by the elementary potential theory formula

$$\mathbb{P}_y(\text{SRW on } \mathbb{Z}^d \text{ ever hits } 0) \leq c_d \|y\|_1^{2-d},$$

for some finite c_d and all y (see [La, Proposition 1.5.9]), we get that $g(x) \leq \kappa_d \|x\|_1^{2-d}$, for some κ_d finite, with condition (1.2) implying that S_* is finite. \square

Proof of Proposition 1.12. By definition of the EGS interaction, necessarily $\mathbb{G}_{\eta_n} = \overline{\mathbb{B}}_k$ for $\eta_n \in [\tau_k, \tau_{k+1})$. To each $k \geq 1$ correspond $L_k \in [1, N(k)]$ such stopping times, and $Z_{\eta_n} \in \mathcal{C}_k$ for all but the smallest of these (namely, $\eta_n = \tau_k$, $k \geq 2$), in which case $Z_{\eta_n} \in \overline{\mathbb{B}}_{k-1}$ is within distance one of $\partial\overline{\mathbb{B}}_{k-1}$. Consequently, S of Lemma 1.3 is bounded above by the LHS of (1.4). Further, if $\sup_z \deg_{\overline{\mathbb{G}}}(z) \leq \overline{\deg}$ finite, then conditional on \mathcal{F}_t^Z , upon each visit of $\partial\overline{\mathbb{B}}_k$ by Z_t (i.e. time σ_n), we have that Z_{t+1} is not in $\partial\overline{\mathbb{B}}_k$ with probability at least $\epsilon := 1/\overline{\deg}$. It follows that the collection $\{L_k\}$ stochastically dominates the independent Binomial($N(k), \epsilon$) variables $\{L'_k\}$, hence S stochastically dominates the LHS of (1.3) with $N(k)$ replaced there by L'_k . The latter is the monotone upward limit T_∞ of a series $T_n = \sum_{k=1}^n q_k L'_k$, with $q_k \in [0, 1]$ non-random and condition (1.3) amounting to $\mathbb{E}T_n \uparrow \infty$. With $\text{var}(T_n) \leq \mathbb{E}T_n$, we have that $T_n/\mathbb{E}T_n \rightarrow 1$ in probability, hence (1.3) yields that a.s. $S \geq T_\infty = \infty$. Our

thesis about the a.s. transience and recurrence of the corresponding EGS thus follows from Lemma 1.3 (we note in passing that the assumption of $\overline{\mathbb{G}}$ transient is only used in part (b) of Lemma 1.3 for dealing with $\partial\mathbb{G}_{\eta_n} = \emptyset$, which can not occur for EGS).

In case of $\overline{\mathbb{G}} = \mathbb{Z}^d$, $d \geq 3$, $c \geq 1$, upon replacing $\overline{\mathbb{B}}_k$ by \mathbb{B}_{ck} it remains only to verify that our conditions (1.3) and (1.4) are equivalent to the divergence, respectively convergence, of $\sum_k N(k)k^{1-d}$. This follows by potential theory, since

$$k^{d-1}\mathbb{P}_x(\text{SRW on } \mathbb{Z}^d \text{ hits 0 before } \partial\mathbb{B}_{ck}), \quad (2.1)$$

is bounded above and below away from zero, uniformly over $k \geq 1$ and $x \in \mathbb{B}_{ck}$ whose graph distance from $\partial\mathbb{B}_{ck}$ is between 1 and $2dc$. We note in passing that having here x within constant distance of $\partial\mathbb{B}_{ck}$, the standard error term turns out to be $O(1)$ (see formula of [La, Proposition 1.5.10]), so for the stated uniform lower bound it must be refined by using asymptotics of Green's function (cf. [LL, Page 96]). In case $d = 2$, the probabilities appearing in (2.1) are similarly bounded below by $C/(k \log k)$ for some $C > 0$, all k and relevant x (see [La, Proposition 1.6.7]; here the error term is refined using the asymptotics of potential kernel, cf. [LL, Page 104]). With $N(k) \geq 1$, it follows that in this case (1.3) holds, yielding the a.s. recurrence of the EGS, in agreement with the divergence of $\sum_k N(k)k^{1-d}$ for $d = 2$. \square

Proof of Proposition 1.15. With $\xi_r := \inf\{t \geq 0 : \mathbb{D}_t \cap \mathbb{B}_r^c \neq \emptyset\}$, denoting the first time the tip of \mathbb{D}_t reaches the sphere of radius r around 0, we construct $L = O(m^d/\log m)$ stopping times $\sigma_1 < \sigma_2 < \dots < \sigma_L$ within the time interval $[\xi_m, \xi_{2m-1}]$, such that for some constant $\delta < 1$,

$$m^{d-1+\delta}\mathbb{P}(R_s = 0 \text{ for some } s \in [\sigma_\ell, \sigma_{\ell+1}) \mid \mathcal{F}_{\sigma_\ell}) \quad (2.2)$$

is bounded away from zero, uniformly in ℓ and m . Similarly to the proof of Lemma 1.3, upon considering the union of these events over all dyadic $m = 2^k$, the a.s. sample path recurrence of the ROBT interacting SRW $\{R_t\}$ then follows by Paul Lévy's extension of Borel-Cantelli. To this end, since the Euclidean ball \mathbb{B} is γ -almost regular for the growing domains \mathbb{D}_t , it follows that for all t large and some non-increasing $f(\cdot) \geq 1$,

$$\mathbb{B}_{f(t)} \subseteq \mathbb{D}_t \subseteq \mathbb{B}_{f(t)+\gamma \log f(t)}. \quad (2.3)$$

We set $w := \gamma \log(2m)$, the maximal fluctuation $\gamma \log f(t)$ in shape of \mathbb{D}_t when $t \leq \xi_{2m-1}$ (hence $f(t) \leq 2m$). From (2.3) one has that $\mathbb{D}_{\xi_{2m-1}} \supseteq \mathbb{B}_{2m-w}$ (as otherwise $2m - w > f \geq 2m - \gamma \log f$ for some $f = f(\xi_{2m-1}) \leq 2m$, contradictory to our choice of w). There are thus at least $C'm^d$ edges in $\mathbb{D}_{\xi_{2m-1}} \setminus \mathbb{D}_{\xi_m}$, for some universal constant $C' > 0$ and all m . Further, domain growth occurs in ROBT only when $R_t \in \partial\mathbb{D}_t$, and each such boundary visit entails adding at most C'/C edges to \mathbb{D}_{t+1} for some universal constant $C > 0$ (see Defn. 1.5). So for each m there are at least Cm^d such boundary visits within $[\xi_m, \xi_{2m-1}]$.

Hereafter we fix $\epsilon > 0$ small, set $w_\epsilon = (1 + 2\epsilon)w$, $L = Cm^d/w_\epsilon$ and consider the stopping times $\{\sigma_\ell, 1 \leq \ell \leq L\}$, with σ_ℓ denoting the $w_\epsilon\ell$ -th smallest $t \geq \xi_m$ such that $R_t \in \partial\mathbb{D}_t$. Turning to prove the stated uniform probability lower bound for the corresponding events per (2.2), fix $\sigma = \sigma_\ell$ and $f = f(\sigma) \in \mathcal{F}_\sigma$. Recall Defn. 1.5 that $d^{\mathbb{D}_\sigma}(R_\sigma, \mathbb{B}_f) \leq w$, hence there exists a path in \mathbb{D}_σ of length at most $w_\epsilon - 1$ leading from R_σ to some specific $x \in \mathbb{B}_f$ such that $d^{\mathbb{B}_f}(x, \partial\mathbb{B}_f) \geq \epsilon w$. Setting $\delta := (1 + 2\epsilon)\gamma \log(2d)$, the event \mathcal{A}_ℓ that the SRW

$\{R_{\sigma+s}, s \geq 0\}$ on $\mathbb{D}_{\sigma+s}$ takes this specific path has probability at least $(2d)^{-w_\epsilon} = (2m)^{-\delta}$. Since $\sigma_{\ell+1} \geq \sigma_\ell + w_\epsilon$ and $\mathbb{B}_f \subseteq \mathbb{D}_{\sigma_\ell}$, the event considered in (2.2) contains the intersection of \mathcal{A}_ℓ and the event that starting at position x the SRW on \mathbb{B}_f visits 0 before reaching $\partial\mathbb{B}_f$. Clearly, $f \geq m - \gamma \log f \geq m - w \geq m/2$ (by (2.3)). Here $x \in \mathbb{B}_f$ is of Euclidean distance at least $C_0 \log m$ from $\partial\mathbb{B}_f$ for some constant $C_0 = C_0(\epsilon, \gamma, d) > 0$, all m and ℓ . Hence, by potential theory, the probability that SRW starting at x visits 0 before $\partial\mathbb{B}_f$, is bounded below by $\kappa_d m^{1-d}$ for some $\kappa_d > 0$, all $d \geq 2$, m and any such x (see (2.1) in case $d \geq 3$, and text following it for how to handle $d = 2$). In conclusion, as claimed, uniformly in ℓ and m ,

$$\mathbb{P}(R_s = 0 \text{ for some } s \in [\sigma_\ell, \sigma_{\ell+1}) \mid \mathcal{F}_{\sigma_\ell}) \geq \kappa_d m^{1-d} \mathbb{P}(\mathcal{A}_\ell \mid \mathcal{F}_{\sigma_\ell}) \geq \kappa_d m^{1-d} (2m)^{-\delta}.$$

□

3. PROOF OF PROPOSITIONS 1.18 AND 1.19

Proof of Proposition 1.18. We call $m \geq 0$ a SUPER-NON-NV time if $\{(-1, 0) + E_m, E_m\}$ is *unvisited* by $\{E_t, t < m\}$. We further couple our FOBT walk $\{E_t\}$ to the SRW $\{R(t)\}$ on \mathbb{Z}^2 , both starting at $(0, 0)$, so that $E_{t+1} - E_t = R(t+1) - R(t)$ except if the edge to the left of E_t is not in \mathbb{D}_t , in which case with probability $1/4$ both walks have the same right/up/down increment, while with probability $1/12$ each, $E_{t+1} - E_t$ is the right/up/down increment, while $R(t+1) - R(t) = (-1, 0)$. Clearly, $t \mapsto (E_t - R(t))_1$ is then non-decreasing. Moreover, independently of $\{E_t, R(t), t \leq m\}$, with probability $1/4$ the value of $(E_t - R(t))_1$ increases by one at each SUPER-NON-NV time m for which the edge to the left of E_m is not in \mathbb{D}_0 . Fixing $\epsilon > 0$, let \mathcal{A}_n denote the event that there exist $n^{3/4-2\epsilon}$ SUPER-NON-NV times $m \in [0, n]$ with the edge to the left of E_m not being in \mathbb{D}_0 . If $\mathbb{P}(\mathcal{A}_n) \geq 1 - Cn^{-1}$ for some C finite, then a.s. $(E_n - R(n))_1 \geq 0.1n^{3/4-2\epsilon}$ for all n large (by Borel-Cantelli lemma it holds along dyadic $n_k = 2^k$, which by monotonicity of $(E_n - R(n))_1$ extends to all n large). Since $n^{-1/2-\epsilon}|R(n)| \rightarrow 0$, taking $\epsilon < 1/12$ yields the stated a.s. sample path transience of $\{E_t\}$.

Adapting [ABGK, Sect. 6], we proceed to show that indeed $\mathbb{P}(\mathcal{A}_n) \geq 1 - Cn^{-1}$ for some C finite and all n . To this end, first analogously to [ABGK, Lemma 6.1], we know that $\inf_n (\log n) \mathbb{P}_{(0,0)}(\mathcal{I}_n) \geq C$, for some C positive and events

$$\mathcal{I}_n := \bigcap_{t \leq n} \left\{ R(t) \notin \{(-1, 0), (0, 0)\} \right\}.$$

Next, fixing n , similarly to [ABGK, Lemma 6.4] we call $m \in [n^{2\epsilon}, n]$ a *tan time* if $R[m - \lfloor n^\epsilon \rfloor, m)$ avoids $\{(-1, 0), (0, 0)\} + R(m)$, and $R[0, m - \lfloor n^\epsilon \rfloor]$ avoids $F + R(m)$, for the funnel

$$F := \{(x, y) : x \geq -1, |y| \leq \log^3(n\sqrt{x+2})\}.$$

Equipped with this modification of tan time, it is easy to adapt the proof of [ABGK, Lemma 6.6], yielding that the number of n^ϵ -separated tan times within $[0, n]$, exceeds $n^{3/4-2\epsilon}$ with probability at least $1 - Cn^{-2}$. Then, following the proof of [ABGK, Lemma 6.9], we deduce that under our coupling, for some C finite, with probability at least $1 - Cn^{-1}$, whenever $m < n$ is a tan time for $\{R(t)\}$, at least one of $[m - n^\epsilon, m]$ must be a SUPER-NON-NV time for $\{E_t\}$. Consequently, with such probability there are at least $n^{3/4-2\epsilon}$ SUPER-NON-NV times $m \in [0, n]$. It thus suffices to show that a uniformly bounded away fraction of these times has edge left of E_m that is not in \mathbb{D}_0 .

(a). We reveal whether each of the i.i.d. Bernoulli(p) edges is in \mathbb{D}_0 or not, only when our FOBT walk $\{E_t\}$ first visits one of the two ends of that edge. Hence, ordering the SUPER-NON-NV times $m_1 < m_2 < \dots$, since each SUPER-NON-NV time avoided both the FOBT walk current position and the lattice site immediately to its left, we have not revealed up to time m_k whether the edge left to E_{m_k} is in \mathbb{D}_0 or not. Thus, the joint law of events {edge left to E_{m_k} is not in \mathbb{D}_0 } stochastically dominates the corresponding i.i.d. Bernoulli($1-p$) variables. It then follows that with \mathbb{P}_p -probability at least $1 - Cn^{-1}$, for more than $(1-p)/2$ of the first $n^{3/4-2\epsilon}$ SUPER-NON-NV times m_k , the edge left of E_{m_k} is not in \mathbb{D}_0 , as claimed.

(b). With $\{E_t, t \leq n\} \subset [-n, n]^2$, taking $\epsilon > 0$ small enough, our assumption that $k^{-r}|\mathbb{D}_0 \cap [-k, k]^2| \rightarrow 0$ for $r < 3/4 - 2\epsilon$ implies that at least half of the edges left to locations of the FOBT walk at the first $n^{3/4-2\epsilon}$ SUPER-NON-NV times, are not in \mathbb{D}_0 , as claimed. \square

Proof of Proposition 1.19.

(a). For any integers $d, L \geq 2$, consider the stretched lattice \mathbb{L} , consisting of vertices

$$\{(y_1, \dots, y_d) \in \mathbb{Z}^d : \text{at least one } y_i \text{ is integer multiple of } L\},$$

and the edges of \mathbb{Z}^d between them (i.e. connecting pairs of vertices from \mathbb{L} whose \mathbb{Z}^d -distance is one). Denoting by $\mathbb{J} := (L\mathbb{Z})^d$ the subset of junction sites in \mathbb{L} and fixing $d \geq 3$, whenever K_t visits a site z of \mathbb{L} for the first time, it dispenses one probe per adjacent closed edge of \mathbb{L} (thereby using at most $(2d-1)$ probes if $z \in \mathbb{J}$ and at most one probe otherwise). The resulting PSRW $\{K_t\}$ has the same law as the SRW $\{B_t\}$ on the fixed graph \mathbb{L} which is transient (having finite effective resistance between 0 and ∞). The number of steps τ_i it takes $\{B_t\}$ to travel from any junction site in \mathbb{J} to one of its neighboring junction sites, are i.i.d. random variables whose mean being precisely the number of steps it takes the SRW on \mathbb{Z} to reach from 0 to $\pm L$. Thus, $\mathbb{E}\tau_1$ is of order L^2 (by diffusivity of the SRW on \mathbb{Z}). Further, during such τ_i steps at most $2dL$ vertices of \mathbb{L} are visited by our PSRW, hence at most $(2d)^2L$ new probes are being used. Denoting by $n(t)$ the number of visits made by $\{K_s, s \leq t\}$ to the subset \mathbb{J} , recall that a.s. $t^{-1}n(t) \rightarrow 1/\mathbb{E}\tau_1$ hence

$$\overline{m}_t \leq t^{-1}(2d)^2L(n(t) + 1),$$

is eventually bounded above by c_d/L for some non-random c_d finite and all L (which for $L \rightarrow \infty$ is made as small as one wishes). In case $d = 2$, we modify the preceding construction by using our probes upon first visit of the PSRW to sites $z \in \mathbb{J}$ *only* for opening its right/up/down adjacent edges in \mathbb{L} . The projection of the resulting PSRW to the subset \mathbb{J} of the stretched lattice \mathbb{L} is then a lazy version of the FOBT interaction model considered in Proposition 1.18 (for $\mathbb{D}_0 = \{0\}$ and having here probability $1 - 1/L$ of returning to the current position before reaching an adjacent junction site). Since by Proposition 1.18 the sample path of this FOBT walk on \mathbb{J} is a.s. transient, the same applies for our PSRW, while by the preceding reasoning \overline{m}_t is made as small as one wishes upon choosing $L \rightarrow \infty$.

As for the stated sample path recurrence, fix $d, M \geq 1$ and the one-dimensional subspace $\mathbb{O} := \mathbb{Z} \times \{0\} \times \dots \times \{0\}$ of \mathbb{Z}^d . Here we use $2M$ probes each time step, placing these on the edges within \mathbb{O} that are adjacent to the currently symmetric open interval $\mathbb{D}_t \subset \mathbb{O}$. The resulting PSRW has $\overline{m}_t = 2M$ as large as we wish and merely follows the path of the recurrent one-dimensional SRW on \mathbb{O} .

(b). Here a probe emitted at time t follows a SRW on \mathbb{D}_t , starting from the current position K_t of our PSRW. Fixing $d \geq 2$, we now opt to release at the first visit of our PSRW to each site $z \in \mathbb{Z}^d$, the \mathcal{F}_t -adapted minimal number of probes $m(t)$ required for opening all $2d$ edges of \mathbb{Z}^d adjacent to z (i.e. $m(t) = \inf\{s \geq 1 : \mathbb{B}^{\mathbb{Z}^d}(z, 1) \subseteq \mathbb{D}_{t+s}\}$). It results with the sample path of $\{K_t\}$ matching that of a SRW on \mathbb{Z}^d , which is thereby a.s. transient when $d \geq 3$ and a.s. recurrent when $d = 2$. Further, the sequence $\{m(t)\}$ of probe counts is stochastically dominated by the i.i.d. variables $\xi_t - 1$, with ξ_1 following the $2d$ coupon collector distribution (i.e. the number of independent, uniform samples from among $2d$ distinct coupons one needs for possessing a complete set). Thus, by the SLLN, almost surely,

$$\limsup_{t \rightarrow \infty} \overline{m}_t \leq \mathbb{E}\xi_1 - 1 = c_d$$

(for $c_d := 2d \sum_{\ell=1}^{2d-1} \ell^{-1}$). The same transience/recurrence holds even when extra M probes are emitted at *each* step of the PSRW (yielding $\overline{m}_t \geq M$ arbitrarily large). \square

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